

CHAPTER 2 PARTIAL FRACTIONS

EXERCISE 8 Page 18

1. Resolve $\frac{12}{x^2-9}$ into partial fractions

$$\frac{12}{x^2-9} \equiv \frac{12}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)} = \frac{A(x-3)+B(x+3)}{(x+3)(x-3)}$$

Hence, $12 = A(x-3) + B(x+3)$

If $x = -3$, $12 = -6A$ from which, $A = 12/-6 = -2$

If $x = 3$, $12 = 6B$ from which, $B = 12/6 = 2$

Hence, $\frac{12}{x^2-9} = \frac{-2}{(x+3)} + \frac{2}{(x-3)} = \frac{2}{(x-3)} - \frac{2}{(x+3)}$

2. Resolve $\frac{4(x-4)}{x^2-2x-3}$ into partial fractions

$$\text{Let } \frac{4(x-4)}{x^2-2x-3} \equiv \frac{4x-16}{(x+1)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-3)} = \frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$$

Hence, $4x - 16 = A(x-3) + B(x+1)$

If $x = -1$, $-20 = -4A$ from which, $A = 5$

If $x = 3$, $12 - 16 = 4B$ from which, $B = -1$

Hence, $\frac{4(x-4)}{x^2-2x-3} = \frac{5}{(x+1)} - \frac{1}{(x-3)}$

3. Resolve $\frac{x^2-3x+6}{x(x-2)(x-1)}$ into partial fractions

$$\text{Let } \frac{x^2-3x+6}{x(x-2)(x-1)} \equiv \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-1)} = \frac{A(x-2)(x-1)+Bx(x-1)+Cx(x-2)}{x(x-2)(x-1)}$$

Hence, $x^2 - 3x + 6 = A(x-2)(x-1) + Bx(x-1) + Cx(x-2)$

If $x = 0$, $6 = A(-2)(-1)$ from which, $6 = 2A$ and $A = 3$

If $x = 2$, $4 - 6 + 6 = B(2)(1)$ from which, $4 = 2B$ and $B = 2$

If $x = 1$, $1 - 3 + 6 = C(1)(-1)$ from which, $4 = -C$ and $C = -4$

Hence,
$$\frac{x^2 - 3x + 6}{x(x-2)(x-1)} = \frac{3}{x} + \frac{2}{(x-2)} - \frac{4}{(x-1)}$$

4. Resolve $\frac{3(2x^2 - 8x - 1)}{(x+4)(x+1)(2x-1)}$ into partial fractions

Let

$$\frac{3(2x^2 - 8x - 1)}{(x+4)(x+1)(2x-1)} \equiv \frac{A}{(x+4)} + \frac{B}{(x+1)} + \frac{C}{(2x-1)} = \frac{A(x+1)(2x-1) + B(x+4)(2x-1) + C(x+4)(x+1)}{(x+4)(x+1)(2x-1)}$$

Hence, $6x^2 - 24x - 3 = A(x+1)(2x-1) + B(x+4)(2x-1) + C(x+4)(x+1)$

If $x = -4$, $96 + 96 - 3 = A(-3)(-9)$ from which, $189 = 27A$ and $A = 7$

If $x = -1$, $6 + 24 - 3 = B(3)(-3)$ from which, $27 = -9B$ and $B = -3$

If $x = 0.5$, $1.5 - 12 - 3 = C(4.5)(1.5)$ from which, $-13.5 = 6.75C$ and $C = -2$

Hence,
$$\frac{3(2x^2 - 8x - 1)}{(x+4)(x+1)(2x-1)} = \frac{7}{(x+4)} - \frac{3}{(x+1)} - \frac{2}{(2x-1)}$$

5. Resolve $\frac{x^2 + 9x + 8}{x^2 + x - 6}$ into partial fractions

Since the numerator is of the same degree as the denominator, division is firstly required.

$$\begin{array}{r} 1 \\ x^2 + x - 6 \overline{) x^2 + 9x + 8} \\ \underline{x^2 + x - 6} \\ 8x + 14 \end{array}$$

Hence, $\frac{x^2 + 9x + 8}{x^2 + x - 6} = 1 + \frac{8x + 14}{x^2 + x - 6}$

Let $\frac{8x + 14}{x^2 + x - 6} = \frac{8x + 14}{(x+3)(x-2)} \equiv \frac{A}{(x+3)} + \frac{B}{(x-2)} = \frac{A(x-2) + B(x+3)}{(x+3)(x-2)}$

Hence, $8x + 14 = A(x-2) + B(x+3)$

If $x = -3$, $-24 + 14 = -5A$ from which, $-10 = -5A$ and $A = 2$

If $x = 2$, $16 + 14 = 5B$ from which, $30 = 5B$ and $B = 6$

Hence,
$$\frac{x^2 + 9x + 8}{x^2 + x - 6} = 1 + \frac{2}{(x+3)} + \frac{6}{(x-2)}$$

6. Resolve $\frac{x^2 - x - 14}{x^2 - 2x - 3}$ into partial fractions

Since the numerator is of the same degree as the denominator, division is firstly required.

$$\begin{array}{r} 1 \\ x^2 - 2x - 3 \overline{) x^2 - x - 14} \\ \underline{x^2 - 2x - 3} \\ x - 11 \end{array}$$

Hence,
$$\frac{x^2 - x - 14}{x^2 - 2x - 3} = 1 + \frac{x - 11}{x^2 - 2x - 3}$$

Let
$$\frac{x - 11}{x^2 - 2x - 3} = \frac{x - 11}{(x+1)(x-3)} \equiv \frac{A}{(x+1)} + \frac{B}{(x-3)} = \frac{A(x-3) + B(x+1)}{(x+1)(x-3)}$$

Hence,
$$x - 11 = A(x - 3) + B(x + 1)$$

If $x = -1$, $-1 - 11 = -4A$ from which, $-12 = -4A$ and $A = 3$

If $x = 3$, $3 - 11 = 4B$ from which, $-8 = 4B$ and $B = -2$

Hence,
$$\frac{x^2 - x - 14}{x^2 - 2x - 3} = 1 + \frac{3}{(x+1)} - \frac{2}{(x-3)}$$

7. Resolve $\frac{3x^3 - 2x^2 - 16x + 20}{(x-2)(x+2)}$ into partial fractions

$$\begin{array}{r} 3x - 2 \\ x^2 - 4 \overline{) 3x^3 - 2x^2 - 16x + 20} \\ \underline{3x^3 - 12x} \\ -2x^2 - 4x + 20 \\ \underline{-2x^2 + 8} \\ -4x + 12 \end{array}$$

Hence,
$$\frac{3x^3 - 2x^2 - 16x + 20}{(x-2)(x+2)} \equiv 3x - 2 + \frac{12 - 4x}{x^2 - 4}$$

$$\text{Let } \frac{12-4x}{x^2-4} = \frac{12-4x}{(x-2)(x+2)} \equiv \frac{A}{(x-2)} + \frac{B}{(x+2)} = \frac{A(x+2)+B(x-2)}{(x-2)(x+2)}$$

$$\text{Hence, } 12-4x = A(x+2) + B(x-2)$$

$$\text{If } x = 2, \quad 4 = 4A \quad \text{from which, } A = 1$$

$$\text{If } x = -2 \quad 20 = -4B \quad \text{from which, } B = -5$$

$$\text{Hence, } \frac{3x^3-2x^2-16x+20}{(x-2)(x+2)} \equiv 3x-2 + \frac{1}{(x-2)} - \frac{5}{(x+2)}$$

EXERCISE 9 Page 20

1. Resolve $\frac{4x-3}{(x+1)^2}$ into partial fractions

$$\text{Let } \frac{4x-3}{(x+1)^2} \equiv \frac{A}{(x+1)} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2}$$

$$\text{Hence, } 4x-3 = A(x+1) + B$$

$$\text{If } x = -1 \quad -7 = B$$

$$\text{Equating } x \text{ coefficients gives: } 4 = A$$

$$\text{Hence, } \frac{4x-3}{(x+1)^2} = \frac{4}{(x+1)} - \frac{7}{(x+1)^2}$$

2. Resolve $\frac{x^2+7x+3}{x^2(x+3)}$ into partial fractions

$$\text{Let } \frac{x^2+7x+3}{x^2(x+3)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+3)} = \frac{A(x)(x+3)+B(x+3)+Cx^2}{x^2(x+3)}$$

$$\text{Hence, } x^2+7x+3 = A(x)(x+3) + B(x+3) + Cx^2$$

$$\text{If } x = 0 \quad 3 = 3B \quad \text{from which, } B = 1$$

$$\text{If } x = -3 \quad 9 - 21 + 3 = 9C \quad \text{i.e. } -9 = 9C \quad \text{from which, } C = -1$$

$$\text{Equating } x^2 \text{ coefficients: } 1 = A + C \quad \text{from which, } A = 2$$

$$\text{Hence, } \frac{x^2+7x+3}{x^2(x+3)} = \frac{2}{x} + \frac{1}{x^2} - \frac{1}{(x+3)}$$

3. Resolve $\frac{5x^2-30x+44}{(x-2)^3}$ into partial fractions

$$\text{Let } \frac{5x^2-30x+44}{(x-2)^3} \equiv \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} = \frac{A(x-2)^2+B(x-2)+C}{(x-2)^3}$$

$$\text{Hence, } 5x^2-30x+44 = A(x-2)^2 + B(x-2) + C$$

$$\text{If } x = 2 \quad 20 - 60 + 44 = C \quad \text{from which, } C = 4$$

Equating x^2 coefficients: $5 = A$

Equating constants: $44 = 4A - 2B + C$ from which, $44 = 20 - 2B + 4$

and $2B = 20 + 4 - 44 = -20$ from which, $B = -10$

Hence,
$$\frac{5x^2 - 30x + 44}{(x-2)^3} \equiv \frac{5}{(x-2)} - \frac{10}{(x-2)^2} + \frac{4}{(x-2)^3}$$

4. Resolve $\frac{18 + 21x - x^2}{(x-5)(x+2)^2}$ into partial fractions

Let
$$\frac{18 + 21x - x^2}{(x-5)(x+2)^2} \equiv \frac{A}{(x-5)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} = \frac{A(x+2)^2 + B(x-5)(x+2) + C(x-5)}{(x-5)(x+2)^2}$$

Hence, $18 + 21x - x^2 = A(x+2)^2 + B(x-5)(x+2) + C(x-5)$

If $x = 5$ $18 + 105 - 25 = 49A$ i.e. $98 = 49A$ from which, $A = 2$

If $x = -2$ $18 - 42 - 4 = -7C$ i.e. $-28 = -7C$ from which, $C = 4$

Equating x^2 coefficients: $-1 = A + B$ from which, $B = -3$

Hence,
$$\frac{18 + 21x - x^2}{(x-5)(x+2)^2} = \frac{2}{(x-5)} - \frac{3}{(x+2)} + \frac{4}{(x+2)^2}$$

EXERCISE 10 Page 21

1. Resolve $\frac{x^2 - x - 13}{(x^2 + 7)(x - 2)}$ into partial fractions

$$\text{Let } \frac{x^2 - x - 13}{(x^2 + 7)(x - 2)} \equiv \frac{Ax + B}{(x^2 + 7)} + \frac{C}{(x - 2)} = \frac{(Ax + B)(x - 2) + C(x^2 + 7)}{(x^2 + 7)(x - 2)}$$

$$\text{Hence, } x^2 - x - 13 = (Ax + B)(x - 2) + C(x^2 + 7)$$

$$\text{If } x = 2, \quad 4 - 2 - 13 = 11C \quad \text{i.e.} \quad -11 = 11C \quad \text{from which, } C = -1$$

$$\text{Equating } x^2 \text{ coefficients: } 1 = A + C \quad \text{from which, } A = 2$$

$$\text{Equating constant terms: } -13 = -2B + 7C = -2B - 7 \quad \text{i.e.} \quad 2B = 13 - 7 = 6 \quad \text{from which, } B = 3$$

$$\text{Hence, } \frac{x^2 - x - 13}{(x^2 + 7)(x - 2)} = \frac{2x + 3}{(x^2 + 7)} - \frac{1}{(x - 2)}$$

2. Resolve $\frac{6x - 5}{(x - 4)(x^2 + 3)}$ into partial fractions

$$\text{Let } \frac{6x - 5}{(x - 4)(x^2 + 3)} \equiv \frac{A}{(x - 4)} + \frac{Bx + C}{(x^2 + 3)} = \frac{A(x^2 + 3) + (Bx + C)(x - 4)}{(x - 4)(x^2 + 3)}$$

$$\text{Hence, } 6x - 5 = A(x^2 + 3) + (Bx + C)(x - 4)$$

$$\text{If } x = 4, \quad 24 - 5 = 19A \quad \text{i.e.} \quad 19 = 19A \quad \text{from which, } A = 1$$

$$\text{Equating } x^2 \text{ coefficients: } 0 = A + B \quad \text{from which, } B = -1$$

$$\text{Equating constant terms: } -5 = 3A - 4C = 3 - 4C \quad \text{i.e.} \quad 4C = 3 + 5 = 8 \quad \text{from which, } C = 2$$

$$\text{Hence, } \frac{6x - 5}{(x - 4)(x^2 + 3)} = \frac{1}{(x - 4)} + \frac{-x + 2}{(x^2 + 3)} = \frac{1}{(x - 4)} + \frac{2 - x}{(x^2 + 3)}$$

3. Resolve $\frac{15 + 5x + 5x^2 - 4x^3}{x^2(x^2 + 5)}$ into partial fractions

$$\text{Let } \frac{15+5x+5x^2-4x^3}{x^2(x^2+5)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{(x^2+5)} = \frac{Ax(x^2+5)+B(x^2+5)+(Cx+D)x^2}{x^2(x^2+5)}$$

$$\text{Hence, } 15+5x+5x^2-4x^3 \equiv Ax(x^2+5)+B(x^2+5)+(Cx+D)x^2$$

$$\text{If } x = 0, \quad 15 = 5B \quad \text{from which, } B = 3$$

$$\text{Equating } x^3 \text{ coefficients: } -4 = A + C \quad (1)$$

$$\text{Equating } x^2 \text{ coefficients: } 5 = B + D \quad \text{i.e. } 5 = 3 + D \quad \text{from which, } D = 2$$

$$\text{Equating } x \text{ coefficients: } 5 = 5A \quad \text{from which, } A = 1$$

$$\text{From equation (1), } -4 = 1 + C \quad \text{from which, } C = -5$$

$$\text{Hence, } \frac{15+5x+5x^2-4x^3}{x^2(x^2+5)} \equiv \frac{1}{x} + \frac{3}{x^2} + \frac{2-5x}{(x^2+5)}$$

4. Resolve $\frac{x^3+4x^2+20x-7}{(x-1)^2(x^2+8)}$ into partial fractions

Let

$$\frac{x^3+4x^2+20x-7}{(x-1)^2(x^2+8)} \equiv \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{(x^2+8)} = \frac{A(x-1)(x^2+8)+B(x^2+8)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+8)}$$

$$\begin{aligned} \text{Hence, } x^3+4x^2+20x-7 &= A(x-1)(x^2+8)+B(x^2+8)+(Cx+D)(x-1)^2 \\ &= A(x-1)(x^2+8)+B(x^2+8)+(Cx+D)(x^2-2x+1) \end{aligned}$$

$$\text{If } x = 1, \quad 1+4+20-7=9B \quad \text{i.e. } 18=9B \quad \text{from which, } B = 2$$

$$\text{Equating } x^3 \text{ coefficients: } 1 = A + C \quad (1')$$

$$\text{Equating } x^2 \text{ coefficients: } 4 = -A + B - 2C + D \quad (2')$$

$$\text{Equating } x \text{ coefficients: } 20 = 8A + C - 2D \quad (3')$$

$$\text{Hence } A + C = 1 \quad (1)$$

$$-A - 2C + D = 2 \quad (2) \quad \text{since } B = 2$$

$$\text{and } 8A + C - 2D = 20 \quad (3)$$

$$2 \times (2) \text{ gives: } -2A - 4C + 2D = 4 \quad (4)$$

$$(3) + (4) \text{ gives: } 6A - 3C = 24 \quad (5)$$

$$3 \times (1) \text{ gives: } 3A + 3C = 3 \quad (6)$$

$$(5) + (6) \text{ gives: } 9A = 27 \text{ from which, } A = 3$$

$$\text{From (1): } 3 + C = 1 \text{ from which, } C = -2$$

$$\text{From (2): } -3 + 4 + D = 2 \text{ from which, } D = 1$$

$$\text{Hence, } \frac{x^3 + 4x^2 + 20x - 7}{(x-1)^2(x^2+8)} = \frac{3}{(x-1)} + \frac{2}{(x-1)^2} + \frac{1-2x}{(x^2+8)}$$

5. When solving the differential equation $\frac{d^2\theta}{dt^2} - 6\frac{d\theta}{dt} - 10\theta = 20 - e^{2t}$ by Laplace transforms, for given boundary conditions, the following expression for $\mathcal{L}\{\theta\}$ results:

$$\mathcal{L}\{\theta\} = \frac{4s^3 - \frac{39}{2}s^2 + 42s - 40}{s(s-2)(s^2-6s+10)}$$

Show that the expression can be resolved into partial fractions to give:

$$\mathcal{L}\{\theta\} = \frac{2}{s} - \frac{1}{2(s-2)} + \frac{5s-3}{2(s^2-6s+10)}$$

$$\begin{aligned} \text{Let } \frac{4s^3 - \frac{39}{2}s^2 + 42s - 40}{s(s-2)(s^2-6s+10)} &\equiv \frac{A}{s} + \frac{B}{(s-2)} + \frac{Cs+D}{(s^2-6s+10)} \\ &= \frac{A(s-2)(s^2-6s+10) + B(s)(s^2-6s+10) + (Cs+D)(s)(s-2)}{s(s-2)(s^2-6s+10)} \end{aligned}$$

$$\begin{aligned} \text{Hence, } 4s^3 - \frac{39}{2}s^2 + 42s - 40 &= A(s-2)(s^2-6s+10) + B(s)(s^2-6s+10) + (Cs+D)(s)(s-2) \\ &= A(s^3 - 8s^2 + 22s - 20) + B(s^3 - 6s^2 + 10s) + (Cs+D)(s^2 - 2s) \end{aligned}$$

$$\text{If } s = 0, \quad -40 = A(-20) \quad \text{from which, } A = 2$$

$$\text{If } s = 2, \quad 32 - 78 + 84 - 40 = B(8 - 24 + 20) \quad \text{i.e. } -2 = 4B \quad \text{from which, } B = -\frac{1}{2}$$

$$\text{Equating } s^3 \text{ coefficients: } 4 = A + B + C \quad \text{i.e. } 4 = 2 - \frac{1}{2} + C \quad \text{from which, } C = \frac{5}{2}$$

$$\text{Equating } s^2 \text{ coefficients: } -\frac{39}{2} = -8A - 6B - 2C + D \quad \text{i.e. } -\frac{39}{2} = -16 + 3 - 5 + D$$

from which, $\mathbf{D} = -\frac{3}{2}$

Hence,
$$\frac{4s^3 - \frac{39}{2}s^2 + 42s - 40}{s(s-2)(s^2-6s+10)} = \frac{2}{s} + \frac{-\frac{1}{2}}{(s-2)} + \frac{\frac{5}{2}s - \frac{3}{2}}{(s^2-6s+10)}$$

i.e.
$$\mathcal{L}\{\theta\} = \frac{2}{s} - \frac{1}{2(s-2)} + \frac{5s-3}{2(s^2-6s+10)}$$
